Stationary states and fractional dynamics in systems with long range interactions

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Abstract. - Dynamics of many-body Hamiltonian systems with long range interactions is studied, in the context of the so called $\alpha-$HMF model. Building on the analogy with the related mean field model, we construct stationary states of the $\alpha-$HMF model for which the spatial organization satisfies a fractional equation. At variance, the microscopic dynamics turns out to be regular and explicitly known. As a consequence, dynamical regularity is achieved at the price of strong spatial complexity, namely a microscopic inhomogeneity which locally displays scale invariance.

The out-of-equilibrium dynamics of many body systems subject to long range couplings defines a fascinating field of investigations, which can potentially impact different domains of applications [1]. In a long range system every constituent is simultaneously solicited by the ensemble of microscopic actors, resulting in a complex dynamical picture. This latter scenario applies to a vast realm of fundamental problems, including gravity and plasma physics, and also extends to a rich variety of cross disciplinary studies [2]. In particular, and with specific emphasis on the peculiar non equilibrium features, long range systems often display a slow relaxation to equilibrium. They are in fact trapped in long-lasting out of equilibrium regimes, termed in the literature Quasi Stationary States (QSS) which bear distinct characteristics, as compared to the corresponding deputed equilibrium configuration. A paradigmatic representative of long range interactions, sharing the mean field viewpoint, is the so called Hamiltonian Mean Field model, which describes the coupled evolution of N rotators, populating the unitary circle and interacting via a cosines like potential. In the limit of infinite system size the discrete HMF model is described by the Vlasov equation for the evolution of the single particle distribution function [3]. This leads to a statistically based treatment, inspired by the seminal work of Lynden-Bell, which revealed the existence of two different classes of QSS, spatially homogeneous or inhomogeneous [4, 5]. More recently, and still with reference to the HMF case study, stationary states have been constructed using a dynamical scheme which exploited the formal analogy with a set of uncoupled pendula [6]. This represented a substantial leap forward in the understanding of the dynamical properties of the QSS in the HMF model, beyond the aforementioned statistical approach. Indeed, it was understood that the microscopic dynamics in the inhomogeneous stationary state is regular and explicitly known, an observation which contributed to shed light onto the puzzling abundance of emerging regular orbits as revealed in [7].

These last results have been though obtained in the framework of mean field systems: The actual distance between particles does no explicitly appear in the HMF potential. In this letter we aim at bridging this gap, by focusing on the so called $\alpha-$HMF model [8]. We shall

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here concentrate on the long range version of the model which implies assuming $0 < \alpha < 1$ [2]. In this long range version the model behaves at equilibrium as the HMF, see for instance [1, 9, 10]. We may then ask if the same correspondence applies to the out of equilibrium dynamics. QSS exists for the $\alpha-$HMF model,as shown in [11]. However, can one still appreciate the asymptotic trend towards regularity? How does the spatial organization impact on the aforementioned features? It was also shown recently that fractional calculus may be a crucial ingredient when dealing with long range systems [12], and we shall see how this point enters the picture in the considered case. To anticipate our findings, we shall here show that all stationary states of the HMF model are shared by the $\alpha-$HMF model: Particles still exhibit regular orbits, at the price of a enhanced microscopic spatial complexity, which materializes in a small scale inhomogeneity being locally scale invariant. Let us start by introducing the governing Hamiltonian which can be cast in the form:

$$H = \sum_{i=1}^{N}\left[\frac{p_i^2}{2} + \frac{1}{2\tilde{N}}\sum_{j\neq i}^{N}\frac{1-\cos(q_i - q_j)}{\|i-j\|^{\alpha}}\right] , \qquad (1)$$

where q_i stands for the orientation of the rotor occupying the lattice position i, while p_i labels the conjugate momentum. The quantity $\|i - j\|$ denotes the shortest distance on the circle of circumference N. The coupling constant between classical rotators decays as a power law of the sites distance. The HMF limit is recovered for $\alpha = 0$. For N even, we have

$$\tilde{N} = \left(\frac{2}{N}\right)^{\alpha} + 2\sum_{i=1}^{N/2-1}\frac{1}{i^{\alpha}} , \qquad (2)$$

which guarantees extensivity of the system. The equations of motion of element i are derived from the above Hamiltonian and can be written as follows

$$\dot{p_i} = -\sin(q_i)C_i + \cos(q_i)S_i = M_i \sin(q_i - \varphi_i) . \qquad (3)$$

where use has been made of the following global quantities:

$$C_i = \frac{1}{\tilde{N}}\sum_{j\neq i}\frac{\cos q_j}{\|i-j\|^{\alpha}} \qquad (4)$$

$$S_i = \frac{1}{\tilde{N}}\sum_{j\neq i}\frac{\sin q_j}{\|i-j\|^{\alpha}} , . \qquad (5)$$

These identify the two components of a non-local magnetization per site, with modulus $M_i = \sqrt{C_i^2 + S_i^2}$, and phase $\varphi_i = \arctan(S_i/C_i)$. In doing so, one brings into evidence the formal analogy with the HMF setting. Indeed we notice that each individual $\alpha-$HMF particle obeys a dynamical equation which closely resembles that of a pendulum. This observation represented the starting point of the analysis carried out in [6], where stationary states were constructed from first principles. More specifically, the authors of [6] imagined that the system of coupled rotators reached a given stationary state, characterized by a constant magnetization in the limit for $N \to \infty$. Then, the HMF model is mapped into a set of N uncoupled pendula, constrained to collectively return a global magnetization identical to that driving their individual dynamics. Self-consistency is hence a crucial ingredient explicitly accommodated for the formulation proposed in [6]. Technically, the stationary state is built by exploiting the ergodic measure on the torus originating from the pendulum motion, a working ansatz that we cannot invoke in the context of the $\alpha-$HMF, due to site localization. Eventually we will overcome this difficulty by considering the continuous limit. We notice that, for large N , and since $0 < \alpha < 1$, the following estimate applies

$$\tilde{N} \approx \frac{2}{1-\alpha}(N/2)^{1-\alpha} . \qquad (6)$$

We can then use expression (6) in Eq.(4) and, as $N \to \infty$, introduce the continuous variables $x = i/N$ and $y = j/N$ and arrive to the following Riemann integral

$$C(x) = \frac{1-\alpha}{2^{\alpha}}\int_{-1/2}^{1/2}\frac{\cos(q(y))}{\|x-y\|^{\alpha}}dy , \qquad (7)$$

where $\|x - y\|$ represents the minimal distance on a circle of circumference one. By invoking the Riemann-Liouville formalism on the circle, we recognize the fractional integral $I^{1-\alpha}$. To be more precise, the Riemann-Liouville formalism on a circle cannot be defined in a consistent way, as the Riemann-Liouville operations cannot map periodic functions into periodic functions. However, the integration is performed over the whole circle in Eq.(7). And, in fact, we are relying on fractional integrals of the potential type, which means we are actually considering a linear combination of Weyl fractional derivatives, see for instance [13] for more details. Consequently we can write

$$C(x) = \frac{1-\alpha}{2^{\alpha}}\Gamma(1-\alpha)I^{1-\alpha}(\cos q(x)) . \qquad (8)$$

A similar relation holds for the $S(x)$ component. Studying the $\alpha-$HMF dynamics implies characterizing the evolution of the scalar fields $q(x, t)$ and $p(x, t)$ which are ruled by the fractional (non-local) partial differential equations

$$\begin{aligned} \frac{\partial q}{\partial t} &= p(x,t) \\ \frac{\partial p}{\partial t} &= \frac{\mu}{2^{\alpha}}\Gamma(\mu)\left(-\sin(q)I^{\mu}(\cos q) + \cos(q)I^{\mu}(\sin q)\right) . \end{aligned}$$

where $\mu = 1 - \alpha$.

At variance with the simpler HMF ($\alpha = 0$) model, the spatial organization $q(x)$ matters within the general setting $0 < \alpha < 1$, an observation which, as anticipated above, poses technical problems to a straightforward extension of the analysis in [6].

In order to carry on the study and mimick the mean field situation, we turn to considering the particular condition where $C(x) = \langle C\rangle = constant$, where $\langle\cdots\rangle$ denotes a spatial average. At the same time, and without losing generality, we require $I^{\mu}(\sin q) = 0$, which corresponds to setting to zero the S component of the magnetization. Hence, we choose to deal with a constant magnetization amount M such that $C(x) = M$.

Note that the translation invariance along the lattice is also likely to statistically lead to the configuration here scrutinized [10]. For finite size systems, the identity $C(x) = M$ can be solely matched by the trivial state where all q are equal. Conversely, in the infinite N limit, the latter assumption, or equivalently $dC/dx = 0$, implies:

$$\mathcal{D}^{\alpha} \cos q = \frac{d^{\alpha} \cos q}{dx^{\alpha}} = 0\,. \tag{9}$$

where we recalled expression (8) and where the operator D^{α} stands for the fractional derivative. Trivial states (q(x)=Cte) as evidenced in the finite size approximation are also solutions of this equation. However, as we will argue in the following, the continuum limit enables us to compute an independent set of admissible solutions. This task is accomplished by exploiting the fact that $\alpha < 1$, which makes the integral $\int 1/\|x\|^{\alpha} dx$ convergent near 0 and that the function $1/\|x-y\|^{\alpha}$ is smooth. Noticing that we rewrite Eq. (7) as:

$$C(x) = \frac{1-\alpha}{2^{\alpha}} \sum_{k=0}^{L-1} \int_{k/L}^{(k+1)/L} \frac{\cos q(y)}{\|x-y\|^{\alpha}}\, dy, \tag{10}$$

using the regularity of $1/\|x-y\|^{\alpha}$, we can extract it from the integral:

$$C(x) \approx \frac{1-\alpha}{2^{\alpha}} \sum_{k=0}^{L-1} \frac{1}{\|x-y_k\|^{\alpha}} \int_{k/L}^{(k+1)/L} \cos q(y)\, dy, \tag{11}$$

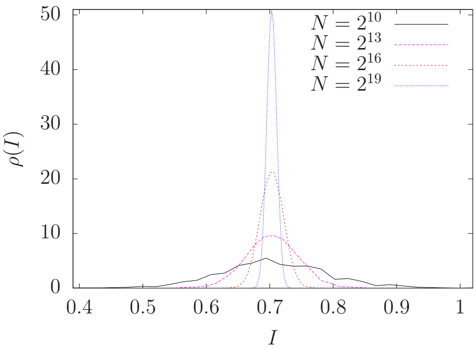
with $y_k \in [k/L,(k+1)/L]$ The above approximation can be rigorously derived via a detailed expansion that we enclose in the annexed appendix. Taking advantage from the latter expression and aiming at reproducing a non trivial state with $C(x) = M$, we now assign the q values on a circle so that in any small interval the average of $\cos q(x)$ is constant and equal to the global, constant magnetization M of the system. This procedure implies a peculiar spatial organization which returns a constant coarse grained image of the function $\cos q$, equal in turn to M. Note also that a similar coarse grained procedure is adopted in [14], with reference to a different case study. In doing so, and as a straightforwad consequence of Eq.(11), which ultimately holds because of the continuum limit hypothesis, we build up a configuration which returns the sought condition: $C(x)$ is a constant function on the circle and equal to the magnetization M. The complex spatial organization of $(p(x), q(x))$ yielding to the constant coarse grained value of the magnetization M, can be in principle condensed into a local (i.e. dependent on x) one particle distribution function $f(p, q, x)$. Clearly, by construction, any configuration $(p(x), q(x))$ compatible with $f(p, q, x)$, is going to be a solution of the fractional Eq.(9)[1] However if we aim at building a configuration that shares some similarities with the aforementioned QSS, we still need stationarity, in the infinite N limit. This property is guaranteed if we refer to the stationary distributions that were calculated for the HMF in [6]. The recipe goes as follows: we pick up q and p values as originating from this stationary distribution, so as to ensure that the time evolution will only consist of a local reshuffling of the actual phase space coordinates, without affecting their associated coarse grained value. In doing so, we hence obtain a family of stationary solutions of the α-HMF model. [2]

The microscopic dynamics of the particles in such states can be mapped into a pendulum motion and is hence integrable. Interestingly, the spatial organization is locally scale free. The functions $q(x,t)$ and $p(x,t)$ are thus "very complicated" along the spatial direction, while displaying a regular time evolution and no chaos. In order to validate this result for a finite size sample on the original lattice we proceed as follows. We consider a stationary state of the HMF consisting of just one torus with associated magnetization $M = 0.5$. In pendulum action-angle variable we refer hence to a one particle distribution of the type $f(I, \theta) = \delta(I - I_0)/2\pi$ (see for instance [17] for details regarding the transformation from (p, q) to (I, θ) and vice versa). We are focusing our attention on an "individual component" belonging to the extended set of a linearly independent elements, which define the QSS basis. In order to be as close as possible to the stationary state of the α-HMF model, we simply distribute randomly on the lattice the values picked from such, analytically accessible, distribution. The analysis for different values of the number of sites and $\alpha = 0.25$, is then performed by monitoring the values of the action and shows as expected a trend towards the stabilization of such a state as N increases (see Fig. 1). We also checked that as expected increasing the value of α, which weakens the coupling strength, implies inducing a more pronounced destabilization of the state, which can be effectively opposed by increasing the number of simulated rotors. The solutions here constructed are hence stable versus the α-HMF dynamics, provided the continuum limit is being performed and so represent a consistent analytical prediction for the existence of quasi-stationary states, beyond the original HMF setting.

To further scrutinize the dynamics of the α-HMF model

[1] Note that x corresponds to a point on the circle for $(p(x), q(x))$, while, for for $f(p, q, x)$, it labels the position x of an interval of width dx.

[2] We note that several distinct stationary distributions of the original HMF model are in principle allowed, yielding to identical values of M. It should be hence possible to build up more complex distributions than those here analyzed, via a combination of such distinct profiles assigned to different spatial locations. A stationary distribution of the α-HMF model can thus correspond to a stationary distribution of the HMF model distributed scale free on the circle. Alternatively, it can be decomposed in a sum of different stationary distributions, which all have the same magnetization and are placed in different regions of the circle. Notice that certain restrictions due to Eq.(9), may be necessarily imposed at the frontiers.



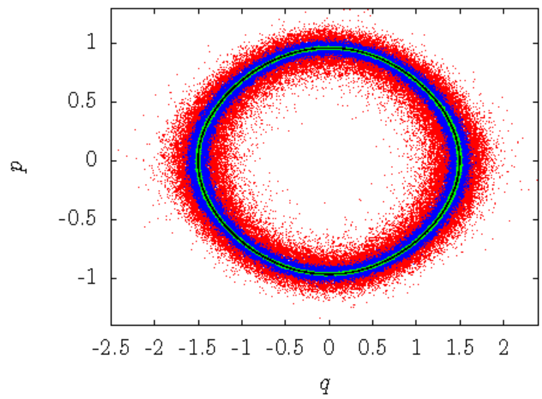


Fig. 1: Top: QSS for the α-HMF model built using a stationary solution of the HMF model corresponding to a one particle PDF of the type $f(I, \theta) = \delta(I - I_0)/2\pi$ (one torus $M = 0.5$). The system is evolved via the α-HMF dynamics with $\alpha = 0.25$. The figure displays the resulting distribution of actions I at $t = 200$, for different sizes. The integration uses the fifth order optimal symplectic integrator [15] and a time step $\delta t = 0.05$. Bottom: A simple superposition of snapshots relative to different values of α and at final time $t = 200$, with $N = 2^{19}$. $\alpha = 0.75$, $\alpha = 0.5$, $\alpha = 0.25$, $\alpha = 0$, correspond respectively to the (online) colors red, blue green and black. Clearly the thickness of the rings get reduced as $\alpha \to 0$, i.e. when the α-HMF model tends towards its corresponding HMF limit.

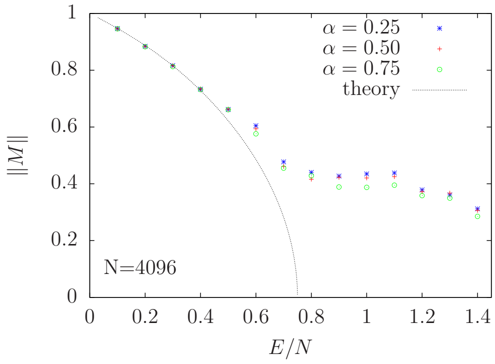


Fig. 2: Magnetization vs. energy per particle E/N for different α, for a system made of $N = 4096$ particles. The magnetization values here reported follow from a time average over the window $400 < T < 800$.

we turn to direct simulations, starting from out of equilibrium initial condition. Our declared goal is to follow the system evolution through the violent relaxation process and eventually identify the presence of spontaneoulsy emerging QSS for the α-HMF model. In particular, we are interested in their microscopic characteristic to make a bridge with the analysis developed in the first part of the paper. For this purpose we initialize the system in $q = 0$ and assume a Gaussian distribution for the conjugate momenta p [9]. The system state is monitored by estimating the average magnetization amount as a function of the energy, see Fig. 2. For energies larger than 0.75 one would expect the homogeneous solution to prevail, as dictated by the statistical mechanics calculation. However, the system gets confined into a inhomogeneous state, the residual time averaged magnetization being large and persistent in time. It is therefore tempting to interpret those states as QSS, and so analyze their associated dynamical features in light of the above conclusions. In particular, we expect the microscopic dynamics to resemble that of a pendulum, bearing some degree of intrinsic regularity. To unravel the phase space characteristics we compute the Poincaré sections, following the recipe in [7, 16] and so visualizing the single particle stroboscopic dynamics, with a rate of acquisition imposed by the self-consistent mean field evolution. The averages of C_i and S_i refer to the two components of the magnetization per site. The Poincaré sections are drawn by recording the positions p_i and $q_i - \varphi_i$ in phase space each time the equality $M_i = M$ is verified. Results for a specific initial conditions are depicted in Fig. 3 where one hundred trajectories are retained. The phase portrait shares many similarities with that obtained for a simple one and a half degree of freedom Hamiltonian (see for instance [17]), with many resonances and invariant tori. Clearly, and in agreement with the above scenario, a large number of particles exhibit regular dynamics. However as the nature of phase space reveals, these QSSs are steady states of the discrete dynamics, not stationary solutions. Nevertheless, we set to analyze the spatial organization of the identified steady state to test whether stationary state features are present in this configuration. To this end, we computed the values of the local magnetization $M(x, t) = \sqrt{C(x, t)^2 + S(x, t)^2}$ versus time and estimated an individual action, stemming from a Hamiltonian pendulum, which would give rise to an equation of motion formally identical to the Eq.(3). Results of the analysis are depicted in Fig. 4. One clearly sees that the function $M(x, t)$ is homogeneous and presents a dependence on t, thus suggesting that the distribution of $q(x, t)$ is a solution of Eq.(9). The plot of action versus time as depicted in Fig. 4, clearly indicates a degree of enhanced spatial complexity, nearby particles not belonging to the same tori. We find in these simulations and in this (N- finite) steady state the same distinctive features of the stationary solutions as depicted earlier.

To conclude, in this Letter we have shown that station-

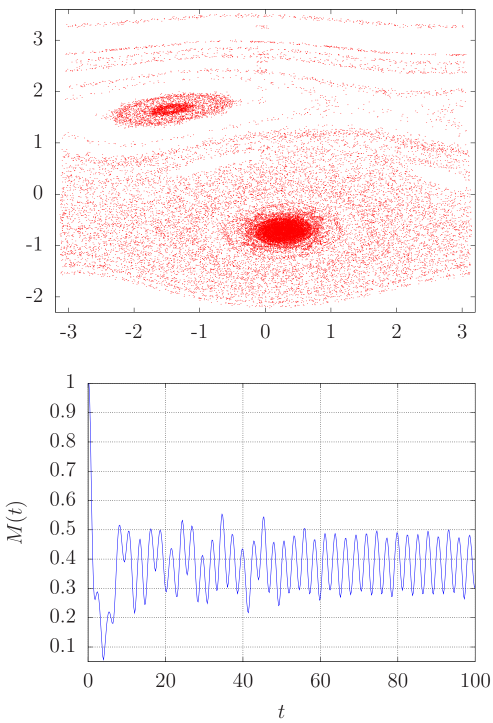


Fig. 3: Top: "Poincaré section" of a QSS for $E/N = 1.2$, $N = 65536$. Initial conditions are Gaussian in p and $q = 0$. The section is computed for $150 < t < 1000$ and $\alpha = 0.25$. Bottom: Magnetization versus time.

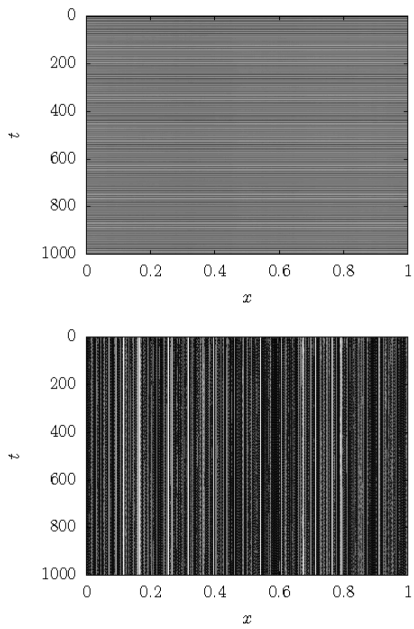


Fig. 4: Top: $M(x) = \sqrt{C(x)^2 + S(x)^2}$ as a function of time ,see also Fig. 3 to monitor to appreciate the amplitude of the fluctuations as recorded along the time evolution. Bottom: individual particle's action as a function of time. The system is initialized to get a QSS with $E/N = 1.2$ and $\alpha = 0.25$ (see Fig. 3). Total number of particle is $N = 8192$, $x = i/N$, with $1 \leq i \leq N$. The actions are those of a pendulum corresponding to Eq.(3). $M(x)$ is almost uniform in space, while the individual spatial organization is complex. The time evolution of the actions appears quite regular.

ary solutions for the mean field HMF model are as well stationary ones of the α-HMF model. Microscopic dynamics in these stationary states is regular, and explicitly known. The price to pay for this microscopic regularity in time is a complex, self similar, spatial organization corresponding to the solution of a fractional equation. When turning to direct numerical investigations we have identified a series of quasi-stationary states, which corresponds to steady states. Still, such states share many of the features of their stationary counterparts. The importance of these conclusions are manifold: On the one side, we confirm that QSS do exist in a generalized non mean field setting [11]. Also, we validate a theoretical method to construct, from first principle, (out of) equilibrium stationary solutions. Finally, the fact that in long range systems stationary states (among which we may of course count the equilibrium) which display regular microscopic dynamics do exist, allows us to dispose of an enormous amount of information regarding the intimate dynamics of a system frozen in such state. This knowledge can prove crucial in bridging the gap between microscopic and macroscopic realms in systems with many body interacting elements. Interestingly, it could prove useful to investigate the role of chaos versus the foundation of statistical mechanics, as discussed in [17, 18].

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Appendix. – As mentioned in the core of the letter, we shall split the integral (7) in L pieces. In this appendix we will justify the procedure that eventually leads to eq.(11). To this end and to make the derivation more transparent we will focus on $C(0)$ and localize the singularity in 0. The generalization for $C(x)$ is straightforward. Let us start by defining the small parameter $\epsilon = 1/L$, and write:

$$C(0) = \frac{1-\alpha}{2^\alpha}\left[\int_0^\epsilon \frac{\cos q(y)}{\|y\|^\alpha} + \sum_{k=1}^{L-1}\int_{k/L}^{(k+1)/L} \frac{\cos q(y)}{\|y\|^\alpha}\,dy\right], \tag{12}$$

Consider the intervals labelled with $k \geq 1$, namely those where the singularity $\|y\| = 0$ is not present. We use the regularity of $1/\|y\|^\alpha$ and perform a Taylor expansion of the function in the vicinity of $y_k = k/L$ to get

$$\frac{1}{\|y\|^\alpha} = \frac{1}{\|y\|^\alpha} - \alpha\frac{y - y_k}{\|y_k\|^{\alpha+1}} + \ldots \tag{13}$$

which implies

$$A_k = \int_{k/L}^{(k+1)/L} \frac{\cos q(y)}{\|y\|^\alpha}\, dy \approx \frac{1}{\|y_k\|^\alpha} \int_{k/L}^{(k+1)/L} \cos q(y) dy - \frac{\alpha}{\|y_k\|^{\alpha+1}} \int_{k/L}^{(k+1)/L} (y - y_k) \cos q(y) dy + \dots .$$

Assume now that the average $\frac{1}{L}\int_{k/L}^{(k+1)/L} \cos q(y)dy$ is constant and equal to M, no matter the value of k. Furthermore, such a property is assumed to hold as L gets larger and larger. Then since $\cos(y)$ is bounded by one, we can get the following estimate

$$|A_k - \frac{1}{\|y_k\|^\alpha} \frac{M}{L}| \lesssim \frac{1}{\|y_k\|^{\alpha+1}} \frac{1}{L^2} . \tag{14}$$

which constraints the A_k term. Exploiting again the fact that $\cos(y)$ is bound and recalling the definition of $C(0)$ we end up with:

$$\begin{aligned} |C(0) - \frac{1-\alpha}{2^\alpha} \frac{M}{L} \sum_{k=1}^{L-1} \frac{1}{\|y_k\|^\alpha}| &\lesssim \epsilon^{1-\alpha} + \frac{1}{L^2} \sum_{k=1}^{L-1} \frac{1}{\|y_k\|^{\alpha+1}} \\ &\lesssim \epsilon^{1-\alpha} + 2\epsilon \int_\epsilon^{1/2} \frac{dy}{y^{\alpha+1}} \\ &\lesssim \epsilon^{1-\alpha} + \epsilon^{1-\alpha} . \end{aligned}$$

This relation confirms the adequacy of the approximated expression (11), as it immediately follows by the definition of M mentioned above. Furthermore, if we take the limit $\epsilon = 1/L \to 0$, we get $C(0) \to M$, a result which is clearly true as long as $\alpha < 1$.

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